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D-branes and deformation quantization

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Abstract: In this note we explain how world-volume geometries of D-branes can be reconstructed within the microscopic framework where D-branes are described through boundary conformal field theory. We extract the (non-commutative) world-volume algebras from the operator product expansions of open string vertex operators. For branes in a flat background with constant non-vanishing B-field, the operator products are computed perturbatively to all orders in the field strength. The resulting series coincides with Kontsevich's presentation of the Moyal product. After extending these considerations to fermionic fields we conclude with some remarks on the generalization of our approach to curved backgrounds.

Keywords: Bosonic Strings, D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

It was observed many years ago that low energy effective actions of (super-)string theories possess solitonic solutions. They are known as solitonic p-branes because of their localization along certain $p+1$-dimensional surfaces in the string-background. Later, Polchinski discovered a remarkable correspondence between such solitonic p-branes and Dp-branes, i.e. boundary conditions for open strings (for a review see [1]). This makes it possible to study branes through the 'microscopic' techniques of boundary conformal field theory.

In the microscopic approach, D-branes are characterized by the way in which they couple to closed string states, i.e. by the 1-point functions of closed string vertex operators. An immediate question is how these couplings relate to the hyper-surfaces one meets in the macroscopic description of D-branes. Here, we shall address a systematic reconstruction of D-brane world-volumes from their world-sheet description. We argue that the information on the geometry of the world-volumes is encoded in the operator products of open string vertex operators. The idea to retrieve geometrical data from operator products is not new, but so far it was mainly applied to closed string vertex operators (see e.g. [2, 3, 4]).

When D-branes are placed in a background which carries a non-vanishing B-field, our procedure will lead us to a deformation of the algebra of functions on the classical world-volume. This phenomenon was first observed by Douglas and Hull [5] in the example of a 2-dimensional brane wrapping a 2-torus (see also [6, 4, 7] for more recent developments in this direction) and it is obviously related to the structure of compactifications of M(atrix) theory on non-commutative tori [8]. Let us remark that non-commutativity enters quite naturally in a theory of open strings. In fact, open string vertex operators are inserted along a one-dimensional line (the boundary of the world-sheet) so that their insertion points are canonically ordered. The product of two open string vertex operators usually depends on the order in which they are inserted and hence it provides an obvious 'source' for non-commutative geometries.

In this short note we restrict our presentation to the case of constant B-fields on a flat background. We shall use standard techniques from conformal perturbation theory (see e.g. [9]) to derive an explicit formula for the non-commutative multiplication in the world-volume algebra. It will turn out as the Moyal deformation of the classical algebra of functions on the world-volume. The deformation depends on the string tension and on the B-field which enters through the anti-symmetric tensor $B(1+B^2)^{-1}$ (see eq. (3.4) below), in agreement with the recent analysis of Chu and Ho [7]. At the end of the text, we extend these considerations to the fermionic sector (see eq. (4.1)).

While some of the formulas below are not new (see e.g. [5, 6]), our techniques are designed for generalizations to non-trivial backgrounds and, in particular, to perturbative studies in the framework of non-linear σ-models where the B-field is then allowed to depend on coordinates. The approach also displays clearly the remarkable relation between open string theories and quantization. In the context of topological open strings this relation is beautifully illustrated by the recent work of Cattaneo and Felder [10] on Kontsevich's quantization formula [11]. Here, we shall see the background metric entering the deformation and the results of topological theories appear only in a very special limit.

2. Open Strings and D-Branes in a Flat Background

To set our stage, we briefly consider the standard bosonic open string moving on a flat d-dimensional Euclidean background, i.e. on the space $\mathbb{R}^d$. Its world-sheet description involves d free bosonic fields $X^i(z,\bar{z})$, $i = 1,\ldots,d$, living on the upper half $\mathrm{Im}z \geq 0$ of the complex plane with Neumann boundary conditions imposed along the boundary $\mathrm{Im}z = 0$. The explicit construction of this field theory is encoded in the formula

$$X^i(z,\bar{z}) = \hat{x}^i - \mathrm{i}\,\frac{\alpha'}{2}\,\hat{p}^i\,\ln z\bar{z} + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{a^i_n}{n}\left(z^{-n}+\bar{z}^{-n}\right)\,, \tag{2.1}$$

where a^i_n, $n<0$ $(n>0)$, create (annihilate) oscillations of the open string and $\hat{x}^i$, $\hat{p}^i$ describe the string's center-of-mass coordinate and momentum. They obey the usual commutation relations involving some constant metric G_{ij} on the background. The parameter α' is the inverse of the string tension, up to some normalization.

From the free bosonic fields $X^i(z,\bar z)$ on the upper half plane we can build many new fields, in particular the chiral currents $J^i(z)$, $\bar J^i(\bar z)$ and open string vertex operators $V_{\rm k}(x)$:

$$
\begin{aligned}
J^i(z) &= 2\mathrm{i}\,\partial X^i(z,\bar z)\\
\bar J^i(\bar z) &= 2\mathrm{i}\,\partial X^i(z,\bar z)\\
V_{\rm k}(x) &= \,:e^{\mathrm{i}\,{\rm k}_i X^i(x)}\,, \qquad (2.2)
\end{aligned}
$$

where $X^i(x) = \hat x^i - \mathrm{i}\,\alpha'\,\hat p^i \ln x + \mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{a^i_n}{n}\,x^{-n}$ is the boundary value $X^i(x)$, $x\in\mathbb{R}$, of the bosonic field $X^i(z,\bar z)$. The open string vertex operators satisfy the following elementary operator product expansions (see e.g. [9]) :

$$
\begin{aligned}
V_{\rm k}(x_1)\,V_{{\rm k}'}(x_2) &\sim \frac{1}{(x_1-x_2)^{\alpha'{\rm kk}'}}\,V_{{\rm k}+{\rm k}'}(x_2)+\cdots \qquad \text{for} \quad x_1 > x_2\\
J^i(z)\,V_{\rm k}(x) &\sim \frac{2\alpha'{\rm k}_i}{z-x}\;V_{\rm k}(x)+\cdots\\
\bar J^i(\bar z)\,V_{\rm k}(x) &\sim \frac{2\alpha'{\rm k}_i}{\bar z-x}\;V_{\rm k}(x)+\cdots\,. \qquad (2.3)
\end{aligned}
$$

Throughout the text we shall raise and lower indices with the help of the metric G and the product ${\rm kk}'$ is defined through ${\rm kk}' = {\rm k}^i{\rm k}'_i$. In all three relations we displayed only the most singular term and represented the rest of the expansion by dots. As we remarked above, operator products of open string vertex operators depend on the ordering of the insertion points, in general. This is why we specified the order $x_1 > x_2$ in the first equation, even though we are presently dealing with a very special situation in which the relation for the other order $x_1 < x_2$ happens to be of the same form.

There exists a more elegant formulation of eqs. (2.3) that makes use of the formal object

$$
f(X(x)) = V[f](x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} d^d{\rm k}\,\hat f({\rm k})\,V_{\rm k}(x)\,.
$$

Here, $\hat f({\rm k})$ denotes the usual Fourier transform of the function $f:\mathbb{R}^d\to\mathbb{C}$. The operator product expansions (2.3) become

$$
\begin{aligned}
V[f\,](1)\,V[g](0) &\sim\; V[f\,g\,](0) \;+\;\cdots\\
J^i(z)\,V[f\,](x) &\sim \frac{1}{\mathrm{i}}\frac{2\alpha' G^{ij}}{z-x}\;V[\partial_j f\,](x)\;+\;\cdots\\
\bar J^i(\bar z)\,V[f\,](x) &\sim \frac{1}{\mathrm{i}}\frac{2\alpha' G^{ij}}{\bar z-x}\;V[\partial_j f\,](x)\;+\;\cdots\,. \qquad (2.4)
\end{aligned}
$$

Note that in the first equation we have specialized to $x_1 = 1, x_2 = 0$. The more general formula in rel. (2.3) can be recovered with the help of conformal transformations so that we did not lose information in passing from eqs. (2.3) to eqs. (2.4).

The first formula in rel. (2.4) is actually quite remarkable since it describes the operator product expansion of open string vertex operators in terms of a very simple algebraic operation, namely in terms of the pointwise multiplication of functions on $\mathbb{R}^d$.

Let us now change the background of our bosonic open string theory by switching on a B-field, i.e. by adding the following term to the action

$$S_B = \frac{1}{4\pi\alpha'} \int_{\mathcal{H}} dz d\bar{z}\, B_{ij}\, J^i(z)\, \bar{J}^j(\bar{z})\,, \tag{2.5}$$

where B_{ij} is some (constant) antisymmetric $d \times d$ matrix and the integral is performed over the upper half plane $\mathcal{H}$. Our main aim is to determine the influence S_B on the operator product of open string vertex operators. At the moment, we shall only look for a suitable formulation of this problem. An explicit solution is then derived in the next section.

It is well known that the only effect of the term (2.5) is to modify the boundary condition along the real line so that the bosonic fields satisfy

$$\partial_y X^i(z,\bar{z}) = B^i{}_j\, \partial_x X^j(z,\bar{z})\,, \qquad \text{where} \qquad z = x + \mathrm{i}y$$

all along the boundary $z = \bar{z}$. The usual Neumann boundary condition $\partial_y X^i = 0$ is recovered in the limit $B_{ij} \to 0$. We could go ahead now and explicitly construct the new theory much as we did for Neumann boundary conditions above (see also [12, 7]). Of course, the formulas for the fields $X^i(z,\bar{z})$ and their boundary values $X^i(x)$ change and so does the operator product expansion of the open string vertex operators, i.e. the latter is no longer described by pointwise multiplication of functions on the background. However, we may still think about the new operator product expansions in terms of functions on $\mathbb{R}^d$, if we are prepared to 'deform' the pointwise multiplication. In other words, we define a new product $(f,g) \mapsto f \star g$ for functions $f, g : \mathbb{R}^d \to \mathbb{R}$ by

$$\left(V[f](1)\, V[g](0)\right)^B \sim V[f \star g](0) \;+\; \cdots\,. \tag{2.6}$$

The superscript B was placed on the left hand side to indicate that the operator product is to be taken in a theory in which the B-field is turned on. The multiplication $\star$ on the right hand side depends on B_{ij} and on the parameter α'.

Even before having constructed $\star$, we can predict some of its properties. The new multiplication will not be commutative, in general. On the other hand, from the sewing constraints of open string theory (see [13, 14, 15]) and the triviality of the fusing matrix in a theory of free bosonic fields it is possible to deduce that $\star$ *is necessarily associative.* Consequently, when functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ are equipped with the product $\star$ they generate a (non-commutative) associative deformation of the algebra of functions on the background. This algebra depends on the anti-symmetric matrix B_{ij} and on the parameter α'.

JHEP06(1999)030

Open strings with Neumann boundary conditions can be interpreted in terms of a D-brane that fills the whole background. We can certainly generalize our considerations to D-branes of smaller dimension. In this case one has to impose Dirichlet boundary conditions in some directions and only those open string vertex operators survive that carry momentum tangential to the brane's world-volume. All constructions in the subsequent sections apply directly to D-branes of smaller dimension. For simplicity of the presentation, we shall stick to the case of a background-filling brane.

3. Perturbation Series for the Deformed Product

To obtain an explicit formula for the product $\star$ defined by rel. (2.6), we regard the term S_B (eq. (2.5)) as a perturbation of the original bosonic theory with Neumann boundary conditions and study the usual field theoretic perturbation expansion. The correlators of the perturbed theory are constructed by the standard prescription

$$\begin{aligned}\langle \Phi_1 \cdots \Phi_N \rangle^B_\epsilon \;&=\; \frac{1}{Z}\, \langle \Phi_1 \cdots \Phi_N \; e^{-S_B} \rangle_\epsilon \\ &:= \frac{1}{Z} \sum_{n=0}^{\infty} \left(\frac{-1}{4\pi\alpha'}\right)^n \frac{1}{n!} \int_{\mathcal{H}^\epsilon_n} d^dz d^d\bar{z} \; \Big\langle \; \Phi_1 \cdots \Phi_N \prod_{a=1}^{n} B_{i_a j_a} J^{i_a}(z_a) \bar{J}^{j_a}(\bar{z}_a) \; \Big\rangle \, .\end{aligned}$$

Here $Z := \langle \exp(-S_B)\rangle_\epsilon$ and the expressions depend on a UV-cutoff ϵ through the domain of integration,

$$\mathcal{H}^\epsilon_n := \Big\{ \, (z_1, \ldots, z_n) \mid \mathrm{Im}(z_a) \, > \, \epsilon \, , \; |z_a - z_b| \, > \, \epsilon \; \text{ for } \; a \neq b \, \Big\} \, .$$

As long as ϵ is positive, the integrals are protected against divergencies caused by the short distance singularities of the integrands.[1] When defining $\mathcal{H}^\epsilon_n$ we assumed that all fields Φ_ν are inserted along the boundary. Eventually, we will choose $\Phi_1 = V[f](1)$ and $\Phi_2 = V[g](0)$.

The correlators in the integrand can be evaluated with the help of Ward identities for chiral currents. There exist two types of terms in these Ward identities. One type involves contractions between two currents while the second appears when we contract currents with one of the fields Φ_ν. Let us first look at contractions between two currents. Their contribution to the deformed correlation functions is evaluated with the help of the operator products expansions

$$J^i(z) \; J^j(w) \sim \frac{2\alpha' G^{ij}}{(z-w)^2} \; + \; \cdots \, , \qquad J^i(z) \; \bar{J}^j(\bar{w}) \; \sim \; \frac{2\alpha' G^{ij}}{(z-\bar{w})^2} \; + \; \cdots \, . \qquad (3.1)$$

Similar expansions exist for $\bar{J}^i$ instead of J^i. We consider a perturbing operator inserted at the point $(z, \bar{z})$ and assume that its anti-holomorphic current $\bar{J}^j$ acts on

[1] IR-divergencies can be cured by introducing a cutoff Λ and restricting the integrations to $|z| < \Lambda$. We refrain from making this more explicit in our formulas.

JHEP06(1999)030

one of the boundary fields while the holomorphic field J^i is contracted with another current (from a perturbing operator at $(w,\bar{w})$) which may be either holomorphic or anti-holomorphic. In both cases, the 2-dimensional integral over the position $(z,\bar{z})$ of the insertion can be converted into a contour-integral. If we pair $J^i(z)$ with another holomorphic current, this contour-integral is easily seen to vanish. A non-trivial contribution arises only when we contract $J^i(z)$ with some anti-holomorphic current $\bar{J}^k(\bar{w})$. After the integration, the two perturbing fields at $(z,\bar{z})$ and $(w,\bar{w})$ turn out to be combined into only one insertion at $(w,\bar{w})$ which has the form:

$$\frac{1}{4\pi\alpha'}\int_{\mathcal{H}} dw d\bar{w}\ \frac{1}{\mathrm{i}}\ B_i{}^j\, B_{jl}\, J^i(w)\, \bar{J}^l(\bar{w})\,.$$

In other words, the fields at $(z,\bar{z})$ have disappeared leaving an insertion at $(w,\bar{w})$ behind which is of the same form as the original expression for S_B but with $\frac{1}{\mathrm{i}}B^2$ appearing instead of B. Iteration of this argument allows us to sum over open chains of current contractions with arbitrary length[2]. With the correct combinatorial factors filled in, the terms form a geometric series. As a result, the perturbing fields are replaced by

$$S_B \ \longrightarrow\ \frac{1}{4\pi\alpha'}\int_{\mathcal{H}} dz d\bar{z}\ \left(\frac{1}{1+\mathrm{i}B}\right)_i^{\ l} B_{lj}\, J^i(z)\, \bar{J}^j(\bar{z}) \tag{3.2}$$

and we are no longer allowed to contract currents among each other. With this simple conclusion in mind we can turn towards the discussion of the second type of contractions in which currents act on one of the boundary fields.

From now on, let us restrict our analysis to the computation of the operator product expansion (2.6). Note that the leading contribution in the operator product expansion of open string vertex operators is completely determined by their 3-point functions. Hence, all our previous remarks on the perturbation of correlation functions apply directly to the perturbation of the operator product. In the following it is quite useful to decompose the matrix in the integral (3.2) into its symmetric and anti-symmetric parts:

$$\frac{B}{1+\mathrm{i}B} = \Theta^{\mathrm{s}} + \Theta^{\mathrm{a}} = -\frac{\mathrm{i}B^2}{1+B^2} \ +\ \frac{B}{1+B^2}\,. \tag{3.3}$$

The symmetric part Θ^{s} gives rise to logarithmic divergencies in the perturbation series for the operator product. They require renormalization of the boundary fields. We shall omit the detailed analysis at this point and only state the result: When the boundary fields $V[f](1)$ and $V[g](0)$ are properly renormalized, there remains no finite contribution coming from Θ^{s}, i.e. the symmetric matrix Θ^{s} is completely absorbed. Hence, after renormalizing the boundary fields, we can work with the

[2]Note that closed loops of current-contractions are canceled by the denominator Z of the deformed correlation functions.

effective perturbing operator

$$S_B^{\text{eff}} = \frac{1}{4\pi\alpha'}\int_{\mathcal{H}} dz d\bar z\ \Theta_{ij}\ J^i(z)\ \bar J^j(\bar z) := \frac{1}{4\pi\alpha'}\int_{\mathcal{H}} dz d\bar z \left(\frac{1}{1+B^2}\right)_i^{\ l} B_{lj}\ J^i(z)\ \bar J^j(\bar z)\,,$$

if we no longer allow for contractions between currents. Here and in the following we drop the upper index ^{a} on the anti-symmetric matrix $\Theta = \Theta^a$.

After all these preparations it is now straightforward to compute the deformed product $f \star g$ to all orders in perturbation theory. One can gain some insight into the structure of this expansion by working out the 0^{th} and 1^{st} order terms in α'.

$$\begin{aligned}
(V[f](1)\,V[g](0))^B &= V[f\,](1)\,V[g\,](0) + \\
&\quad + \frac{\alpha'}{\pi}\int_{\mathcal{H}} dz d\bar z \frac{1}{z-1}\frac{1}{\bar z}\ \Theta^{ij}\ V[\partial_i f\,](1)\,V[\partial_j g\,](0) + \\
&\quad + \frac{\alpha'}{\pi}\int_{\mathcal{H}} dz d\bar z\ \frac{1}{\bar z-1}\frac{1}{z}\ \Theta^{ij}\ V[\partial_j f\,](1)\,V[\partial_i g\,](0) + O((\alpha')^2) \\
&= V[f\,](1)\,V[g\,](0) + \frac{\alpha'}{\pi}\int_{\mathcal{H}} dz d\bar z \left(\frac{1}{z-1}\frac{1}{\bar z} - \frac{1}{\bar z-1}\frac{1}{z}\right) \times \\
&\quad \times \Big(\Theta^{ij}\ V[\partial_i f\,](1)\,V[\partial_j g\,](0)\Big) + O((\alpha')^2)\,.
\end{aligned}$$

Before we comment on this calculation let us state the full result to all orders in α'.

$$\begin{aligned}
(V[f](1)V[g](0))^B &= \sum_n \left(\frac{\alpha'}{\pi}\right)^n \frac{1}{n!}\int dz_1 d\bar z_1 \ldots dz_n d\bar z_n \prod_{a=1}^n \left(\frac{1}{z_a-1}\frac{1}{\bar z_a} - \frac{1}{\bar z_a-1}\frac{1}{z_a}\right) \times \\
&\quad \times \Theta^{i_1 j_1}\cdots\Theta^{i_n j_n} V[\partial_{i_1}\cdots\partial_{i_n} f\,](1)\ V[\partial_{j_1}\cdots\partial_{j_n} g\,](0) \\
&\sim \sum_n \left(\frac{\alpha'}{\pi}\right)^n \frac{1}{n!}\int dz_1 d\bar z_1 \ldots dz_n d\bar z_n \prod_{a=1}^n \left(\frac{1}{z_a-1}\frac{1}{\bar z_a} - \frac{1}{\bar z_a-1}\frac{1}{z_a}\right) \times \\
&\quad \times \Theta^{i_1 j_1}\cdots\Theta^{i_n j_n} V[\partial_{i_1}\cdots\partial_{i_n} f\ \partial_{j_1}\cdots\partial_{j_n} g\,](0)\ +\ \cdots\,.
\end{aligned}$$

In the last step we have kept only the leading contribution from the operator product expansion for vertex operators at $B = 0$. All other manipulations were exact. To understand the expression for the n^{th} summand, it suffices to look at the first order terms. Note that the first order contribution to the operator product contains two terms. One of them arises when J^i acts on the open string vertex operator at the point $x = 1$ and $\bar J^j$ acts on the vertex operator at $x = 0$. To produce the other one needs to interchange the role of J^i and $\bar J^j$. There are certainly terms where both currents act on the same vertex operator. These terms vanish since they involve a contraction of $\partial_i\partial_j$ with the antisymmetric matrix Θ^{ij}.

We could certainly compute the remaining integrals in the previous expression for operator product expansion. But we shall leave them in their present form and produce a more compact answer by introducing the following shorthand notations:

$$\begin{aligned}
w_n &:= \frac{1}{(2\pi)^{2n}}\frac{1}{n!}\int d^n z d^n \bar z \prod_{a=1}^n \left(\frac{1}{z_a-1}\frac{1}{\bar z_a} - \frac{1}{\bar z_a-1}\frac{1}{z_a}\right)\,, \\
\mathrm{B}_n(f,g) &:= \sum \Theta^{i_1 j_1}\cdots\Theta^{i_n j_n}\ \partial_{i_1}\cdots\partial_{i_n} f\ \partial_{j_1}\cdots\partial_{j_n} g\,.
\end{aligned}$$

Our perturbative construction of the operator product for open string vertex operators provides us with an explicit formula for the product $f \star g$ defined through eq. (2.6). It is given by

$$f \star g = \sum_n (4\pi\alpha')^n \, w_n \, \mathrm{B}_n(f,g) \, . \tag{3.4}$$

An explicit computation of the weight factors gives $w_n = 1/n!$ so that we can perform the summation to bring our result into the much simpler form

$$(f \star g)(\mathrm{x}) = \exp(4\pi\alpha'\Theta^{ij}\partial_i\partial'_j)\, f(\mathrm{x})\, g(\mathrm{x}')|_{\mathrm{x}=\mathrm{x}'} \, .$$

Here, ∂_i and ∂'_j denote differentiation with respect to x_i and x'_j, respectively. Our second expression for $f \star g$ is known to provide a *Moyal product* of the functions f, g. The first formula (3.4) for $\star$ coincides with Kontsevich's formula for the Moyal product of functions on the background $\mathbb{R}^d$ [11]. When one generalizes the above analysis of operator product expansions to models with a non-constant B-field, the answer appears in the form an infinite sum, much as in eq. (3.4).[3] This was our motivation to display the more complicated formula first before computing the weight factors w_n.

The remarks above guarantee that $\star$ is associative but one may also check this directly now. On the other hand, it is not commutative as we infer e.g. from the first order term in the commutator:

$$f \star g - g \star f = 8\pi\,\alpha'\,\Theta^{ij}\;\partial_i f\;\partial_j g \;+\; O((\alpha')^2) = 8\pi\,\alpha'\,\{\,f\,,\,g\,\}_\Theta \;+\; O((\alpha')^2)\,,$$

where we have introduced the Poisson bracket $\{\,.\,,\,.\,\}_\Theta$ through the second equality. From this we conclude that operator products of boundary fields give rise to the usual Moyal deformation in the direction of the 'Poisson bi-vector'

$$\Theta = B\;(1+B^2)^{-1} \, . \tag{3.5}$$

Note that $\Theta \sim B$ for small magnetic fields while $\Theta \sim B^{-1}$ when B^2 becomes very large. We shall refer to the region of very strong fields as 'topological regime'. In this limit, the metric G can be neglected in comparison to the B-field. Our results for strong fields are consistent with the recent analysis [10] of topological σ-models which predicts the deformation for large B^2 to be in the direction of B^{-1}. The latter is the 'Poisson bi-vector' associated with the 'symplectic form' B. Let us also remark that the expression (3.5) is known from the theory of toroidal compactifications where it appears as the B-field on the dual torus.

[3] In the more general case, however, the summation index runs over a set of graphs rather than the integers $n = 0, 1, \ldots$

JHEP06(1999)030

Before we conclude this section, we would like to discuss briefly how the boundary currents $J^i(x) = \bar{J}^i(x)$ of the original field theory make their appearance in the world-volume geometry. With our previous experience on the perturbative computation of operator products, it is rather easy to see that

$$\Big(J^i(1)\ V[f](0)\Big)^B \sim \frac{2\alpha'}{\mathrm{i}}\, G^{ij}\ \left(\frac{1}{1+B^2}\right)_j^{\ k}\ V[\partial_k f](0)\ +\ \cdots .$$

Hence, the chiral boundary currents describe infinitesimal symmetries of the D-brane, i.e. its world-volume algebra inherits derivations δ^i of the form

$$\delta^i = G^{ij}\ \Big((1+B^2)^{-1}\Big)_j^{\ k}\ \partial_k\,. \tag{3.6}$$

In the limit of vanishing B-field, δ^i coincides with the usual derivative $G^{ij}\partial_j$. As one would expect, the number of infinitesimal translational symmetries of a D-brane does not depend on the field strength as long as the background field is constant.

4. Extension to Fermionic Fields

In this final section we would like to extend our analysis to fermionic fields. We consider d Majorana fermions described in terms of the usual (anti-)holomorphic components $\psi^i(z), \bar{\psi}^i(\bar{z})$. Our discussion below focuses on the Neveu-Schwarz sector. For fermions in the Ramond sector the following considerations are easily adjusted and one ends up with a module of the non-commutative algebra that we shall obtain from the Neveu-Schwarz sector.

Recall that the operator product expansions for the fermionic fields are of the form

$$\psi^i(z)\ \psi^j(w) \sim \frac{G^{ij}}{z-w}\ +\ \cdots, \qquad \psi^i(z)\ \bar{\psi}^j(\bar{w})\ \sim\ \frac{G^{ij}}{z-\bar{w}}\ +\ \cdots .$$

Similar expressions hold when holomorphic fields are replaced by anti-holomorphic ones (and vice versa). This motivates to assign the fermionic fields $\psi^i(x) = V[\eta^i](x)$ to generators η^i of a Clifford algebra with multiplication

$$[\,\eta^i\,,\,\eta^j\,]_+ = G^{ij}\,.$$

In this way we may re-express the operator product expansions of fermionic boundary fields in the form $V[\eta^i](1)V[\eta^j](0) = V[[\eta^i,\eta^j]_+](0) = V[\,G^{ij}](0) = G^{ij}$.

Placing the d Majorana fermions in a non-vanishing B-field means to perturb the original theory by the term

$$S'_B = \frac{1}{8\pi}\int_{\mathcal{H}} dzd\bar{z}\ B_{ij}\,\Big(\psi^i(z)\,\partial\psi^j(z) - \bar{\psi}^i(\bar{z})\,\partial\bar{\psi}^j(\bar{z})\Big)\ .$$

When combined with the bosonic sector, the contribution S'_B ensures supersummetry of the total theory with boundary conditions $\psi^i_- = \mathrm{i}B^i{}_j\psi^j_+$ in the fermionic sector. Here we have introduced $\psi^k_\pm = \psi^k \pm \bar{\psi}^k$.

Our evaluation of the perturbation expansion proceeds very much as in the case of bosonic fields. Once more, contributions from chains of contractions between fields in the perturbing operator can be summed up. This is achieved my means of the standard formula

$$\int dz d\bar{z}\ f(z)\ \bar{\partial}_z \frac{1}{z-w} = -2\pi\mathrm{i}\, f(w)$$

and a similar expression with the role of holomorphic and anti-holomorphic variables interchanged. The procedure allows us to replace the B-field in S'_B by $B\,(1+iB)^{-1}$, if at the same time we refrain from further contractions between the effective insertions.

The aim now is to compute the leading term in the operator product expansion of $\psi^i(1)$ and $\psi^j(0)$. To evaluate the contractions between the 'effective' perturbing field and the boundary operators use the decomposition (3.3) of $B(1+\mathrm{i}B)^{-1}$ into its symmetric and anti-symmetric parts. Because of the anti-commutativity of the fermionic fields, only the symmetric matrix Θ^s enters into the final formula for the deformed operator product:

$$\Big(\psi^i(1)\ \psi^j(0)\Big)^B \sim G^{ij} + G^{ik}\left(\frac{-B^2}{1+B^2}\right)_k^{\ j} \ +\ \cdots = G^{ik}\left(\frac{1}{1+B^2}\right)_k^{\ j} \ +\ \cdots .$$

We can translate this result into a deformation of the Clifford algebra. Under the influence of the B-field, the original generators η^i obey the deformed relations

$$[\ \eta^i \ \stackrel{*}{,}\ \eta^j\ ]_+ = G^{ik}\left(\frac{1}{1+B^2}\right)_k^{\ j} . \tag{4.1}$$

The combination on the right hand side appeared already in eq. (3.6) for the derivations of the world-volume algebra.

The model we have studied possesses an $N=1$ supersymmetry on the world-sheet but our considerations certainly generalize to theories with more supersymmetries. As in the case of closed strings [2, 3], the non-commutative world-volume geometries inherit Dirac operators from the supersymmetry generators of the field theory. In the context of non-commutative geometry [16], they give rise to differential geometries on the branes' world-volumes.

5. Conclusions

In this text we have constructed the world-volume geometry (eqs. (3.4), (4.1)) of D-branes in flat backgrounds. Note, however, that both the underlying concepts and the proposed techniques can be applied beyond these simple examples. In particular, it is possible to study the perturbative expansion for the operator product of open string vertex operators in non-linear σ-models. To treat the dependence of B-fields on coordinates of the background one makes use of standard background field methods [17]. It is quite remarkable that the resulting expansions are still organized very much as in the corresponding version of Kontsevich's quantization formula. We shall report on these issues in a forthcoming paper.

More concretely, one may try to reconstruct the world-volume of D-branes for some specific (CFT-) backgrounds, such as e.g. the WZW-model. The classical world-volume of branes on group manifolds is given by certain 'integer' conjugacy classes of the group [18] and the branes come equipped with a B-field. In case of the SU(2)-WZW theory, their world-volumes are described by fuzzy spheres, at least in an appropriate limit [19].

It might also be interesting to study effective field theories on general D-brane world-volumes within the presented framework. The construction of the effective actions is a problem in boundary perturbation theory. Since the perturbing boundary operators and the 'generators' of the world-volume algebra appear on equal footing, it should be possible to develop a rather general approach towards effective theories on non-commutative spaces.

The techniques of [20] provide another step in this direction. It was shown there that non-commutativity of the boundary operator product expansion is the only obstruction in boundary deformation theory. In this sense, a detailed knowledge about operator products of boundary fields (and about their non-commutativity) is essential for our understanding of general boundary flows and of D-brane moduli spaces, in particular.

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